Purifications $\xrightarrow{}$ Part of the "Church of the larger Hilbert space"

Consider a generic mixed state ρ_A of dimension d_A

$\exists$ a state $|\psi_{AB}\rangle$ such that $Tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|) = \rho_A$

To see why this always is true, note that

We can always write it in terms of its eigen-decomposition

$$\rho_A = \sum_{k=0}^{d_A-1} \lambda_k |c_k\rangle\langle c_k|$$

We can purify this as

$$|\psi_{AB}\rangle = \sum_k \sqrt{\lambda_k}\, |c_k\rangle_A \otimes |k\rangle_B$$

Check that this gives

$$Tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|) = Tr_B\Big(\sum_{kk'} \sqrt{\lambda_k}\sqrt{\lambda_{k'}}\, |c_k k\rangle\langle c_{k'} k'|\Big)$$

$$= \sum_{kk'} \sqrt{\lambda_k}\sqrt{\lambda_{k'}}\, |c_k\rangle\langle c_{k'}|\, \overbrace{Tr(|k\rangle\langle k'|)}^{\delta_{kk'}}$$

$$= \sum_k \lambda_k |c_k\rangle\langle c_k|$$

This makes intuitive sense, can read off $|\psi_{AB}\rangle$ that with probability λ_k, A is in $|c_k\rangle$ & B is in $|k\rangle$

<u>But</u> if we have no way of measuring B

The we don't know what state B is in, so just have that A is in k with prob. λ_k.
But crucially it's a pure state. No randomness was required to prepare it. Or, equivalently, there is a measurement to check with certainty that the system is in the state - namely projection onto $|\psi_{AB}\rangle\langle\psi_{AB}|$

In fact purifications are not unique
There are an infinite no. of purifications of any given state.
This is because we can apply an arbitrary unitary on the B part of $|\psi_{AB}\rangle$ & this does not change the reduced state on A.

$$|\Phi_{AB}\rangle = (I_A \otimes U)|\psi_{AB}\rangle$$

for any U

Check this works:

$$\mathrm{Tr}_B(|\Phi_{AB}\rangle\langle\Phi_{AB}|) = \mathrm{Tr}_B\left((I\otimes U)|\psi_{AB}\rangle\langle\psi_{AB}|(I\otimes U^\dagger)\right)$$

$$= \sum_{kk'} \sqrt{\lambda_k}\sqrt{\lambda_{k'}}\, |c_k\rangle\langle c_{k'}|\, \mathrm{Tr}(\cancel{U|k\rangle\langle k'|U^\dagger})$$

$\delta_{kk'}$

$$= \sum_k \lambda_k |c_k\rangle\langle c_k|$$

How large does the purifying system need to be?

Say you have $\rho = \sum_{i=1}^{K} p_i |x_i\rangle\langle x_i|$?

You could purify this as:

$$|\phi\rangle = \sum_{i=1}^{K} \sqrt{p_i}\, |x_i\rangle |e_i\rangle$$

But if $K > d_A$ then this isn't the most efficient method. Because could purify using a d_λ dimensional environment via the eigendecomposition.

no. of non zero eigenvalues.

Let's end on some fun philosophizing...

This way of [illegible] things starting with real

This way of approaching things starting with mixed states & then discussing purifications suggests probabilities are fundamental & purifications are a useful tool.

But one could alternatively argue (as is more common in most introductions to quantum) that the pure state is more fundamental & the reduced mixed state is a way of dealing with incomplete knowledge.

Is there a distinction between:

1) Tossing a coin and preparing a qubit in $|0\rangle$ if heads & in $|1\rangle$ if tails

2) Preparing a pair of quantum coins in a Bell state $\underbrace{\tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)}_{|\phi_{AB}\rangle}$ & then losing a qubit?

In case 1 we have $\rho = I/2$

2 $\rho_A = \mathrm{Tr}_B\left(|\phi\rangle_{AB}\langle\phi|_{AB} \right) = I/2$

Information theoretically the states are identical. No measurement could tell them apart.

But conceptually they are different. To capture this distinction the terms "proper" and "improper" mixtures were introduced.

"Proper" mixture = mixture due to ignorance of underlying state

"Improper" mixture = mixture arising from studying a subsystem of a composite system.

Fun thought - is there actually a distinction? Are all mixtures in fact improper? or all proper?

Will come back to this again in later weeks when we discuss the measurement problem....